Mean-Field limit of the Bose-Hubbard model in high dimension

Denis Périce
dperice@constructor.university

Joint work with:
Shahnaz Farhat & Sören Petrat

CONSTRUCTOR UNIVERSITY

Motivation

Study: large system of quantum bosons

Usually [3]: many-body $N \to \infty$ mean field:

$$H_N := \sum_{i=1}^{N} (-\Delta_i) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} w(X_i - X_j) \quad \text{acting on } L^2(\mathbb{R}^d, \mathbb{C})^{\otimes_+ N}$$

Statistical description of the interaction for a mean particle $\varphi \in L^2(\mathbb{R}^d)$:

$$h^{\varphi}_{\text{Hartree}} = -\Delta + |\varphi|^2 \star w$$

Bose-Hubbard model: interacting bosons on a lattice

- Great success in physics: Mott-insulator \ Superfluid phase transition, experimental observation [2] & theoretical description of the mean field theory [1]
- Mean field justified when $d \to \infty$ and effective in $d = 3$
- Simple mathematical description

Goals:

- Mean field limit as $d \to \infty$ of the dynamics and the ground state energy
- Describe a phase transition
- Strong and local particle interactions

Bose-Hubbard model

Lattice: $\Lambda := (\mathbb{Z}/L\mathbb{Z})^d$ with $d, L \in \mathbb{N}$ such that $d, L \geqslant 2$ of volume $|\Lambda| = L^d$

One-lattice-site Hilbert space: $\ell^2(\mathbb{C})$ of canonical basis

$$|n\rangle := (0, \dots, 0, \underbrace{1}_{n^{th}\text{index}}, 0, \dots), \; n \in \mathbb{N}$$

2^{nd} **quantization:** creation and annihilation operators:

$$a\,|0\rangle := 0, \quad \forall n \in \mathbb{N}^*, \; a\,|n\rangle := \sqrt{n}\,|n-1\rangle\,,$$
$$\forall n \in \mathbb{N}, \; a^\dagger\,|n\rangle := \sqrt{n+1}\,|n+1\rangle$$
$$[a, a^\dagger] = \mathbb{1}_{\ell^2} \tag{CCR}$$

Particle number: $\mathcal{N} := a^\dagger a$

Fock space: $\quad \ell^2(\mathbb{C})^{\otimes|\Lambda|} \cong \mathcal{F}_+\left(L^2(\Lambda, \mathbb{C})\right) := \bigoplus_{n \in \mathbb{N}} L^2(\Lambda, \mathbb{C})^{\otimes_+ n}$

Bose-Hubbard hamiltonian of parameters $J, \mu, U \in \mathbb{R}$:

$$H_d := -\frac{J}{2d} \overbrace{\sum_{\substack{x,y \in \lambda \\ x \sim y}} a_x^\dagger a_y}^{\mathcal{O}(2d|\Lambda|)} + (J - \mu) \sum_{x \in \Lambda} \mathcal{N}_x + \frac{U}{2} \sum_{x \in \Lambda} \mathcal{N}_x(\mathcal{N}_x - 1)$$

Dynamics for $\gamma_d \in L^\infty\left(\mathbb{R}_+, S^1\left(\ell^2(\mathbb{C})^{\otimes|\Lambda|}\right)\right)$:

$$i\partial_t \gamma_d(t) = [H_d, \gamma_d(t)] \tag{B-H}$$

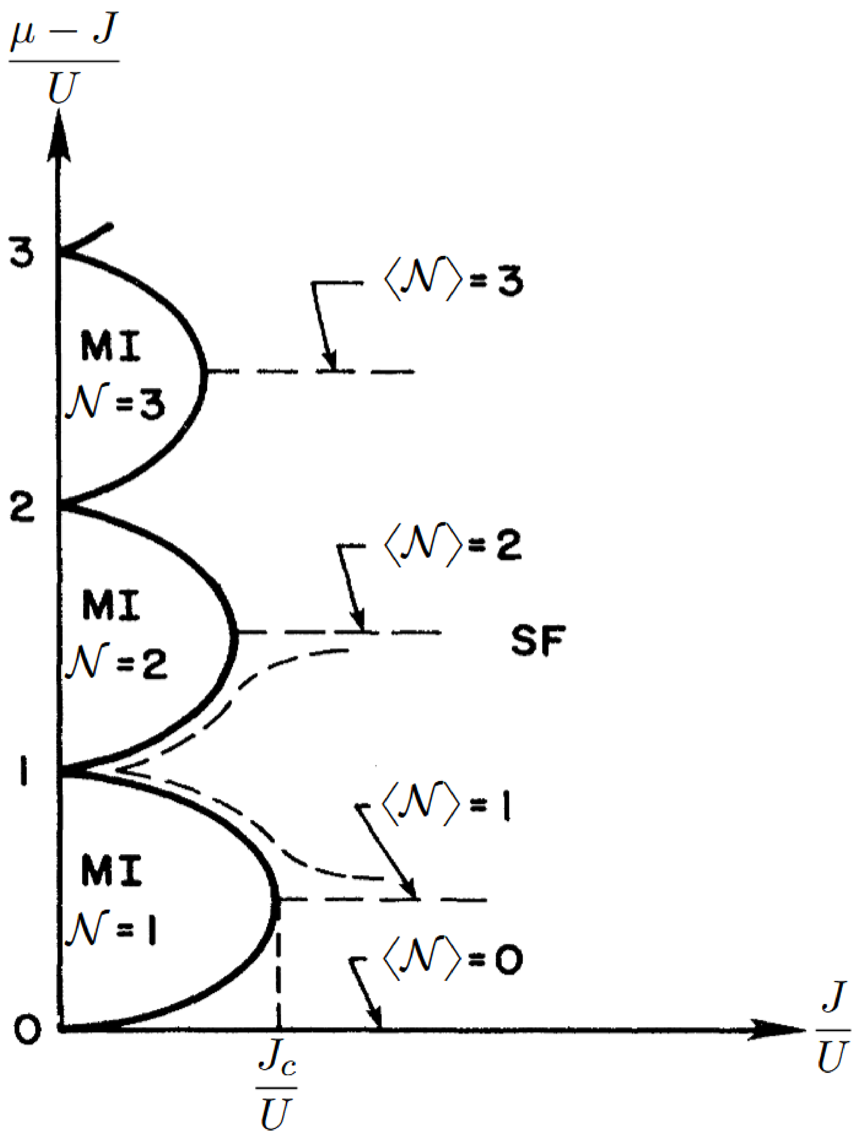
First reduced one-lattice-site density matrix:

$$\gamma_d^{(1)} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathrm{Tr}_{\Lambda \setminus \{x\}}(\gamma_d)$$

Mean field theory

Mean field hamiltonian for $\varphi \in \ell^2(\mathbb{C})$:

$$h^{\varphi} := -J\left(\overline{\alpha_\varphi} a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2\right) + (J - \mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N} - 1)$$



Phase diagram obtained [1] by minimizing $\varphi \mapsto \langle \varphi | h^\varphi \varphi \rangle$

with the order parameter

$$\alpha_\varphi := \langle \varphi | a \varphi \rangle$$

Phase transition: decompose

$$\varphi =: \sum_{n \in \mathbb{N}} \lambda_n \,|n\rangle$$
$$\implies \alpha_\varphi = \sum_{n \in N} \sqrt{n+1}\, \overline{\lambda_n} \lambda_{n+1}$$

- Mott Insulator (MI): $\alpha_\varphi = 0$
- Superfluid (SF): $\alpha_\varphi > 0$

Dynamics:

For $\varphi \in L^\infty\left(\mathbb{R}_+, \ell^2(\mathbb{C})\right)$,

$$i\partial_t \varphi(t) = h^{\varphi(t)} \varphi(t) \tag{mf}$$

Corresponding projection

$$p_\varphi := |\varphi\rangle \langle\varphi|$$

Results

Theorem: *S.Farhat D.P S.Petrat 2025*

Dynamics [4]: Assume

- γ_d solves (B-H) with $\gamma_d(0) \in S^1\left(\ell^2(\mathbb{C})^{\otimes|\Lambda|}\right)$ such that $\mathrm{Tr}\,(\gamma_d(0)) = 1$
- φ solves (mf) with $\varphi(0) \in \ell^2(\mathbb{C})$ such that $\|\varphi\|_{\ell^2} = 1$
- $\exists c_1, c_2 > 0$ such that $\forall n \in \mathbb{N}$,

$$\mathrm{Tr}\,(p_\varphi(0) \mathbb{1}_{\mathcal{N}=n}) \leqslant c_1 e^{-\frac{n}{c_2}}, \quad \mathrm{Tr}\left(\gamma_d^{(1)}(0) \mathbb{1}_{\mathcal{N}=n}\right) \leqslant c_1 e^{-\frac{n}{c_2}}.$$

Then $\exists\, C := C\left(J, c_1, c_2, \mathrm{Tr}\,(p_\varphi(0)\mathcal{N})\right) > 0$ such that $\forall t \in \mathbb{R}_+$,

$$\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{S^1} \leqslant C\left(\left\|\gamma_d^{(1)}(0) - p_\varphi(0)\right\|_{S^1} + \frac{1}{d\sqrt{\ln(d)}}\right) e^{Cte^{Ct}\sqrt{\ln(d)}}$$

Ground state (WIP): If $J, \mu, U \geqslant 0$, then for d large enough,

$$-\frac{\ln(d)^3}{d} \lesssim \inf_{\substack{\psi_d \in \ell^2(\mathbb{C})^{\otimes|\Lambda|} \\ \|\psi_d\| = 1}} \frac{\langle \psi_d | H_d \psi_d \rangle}{|\Lambda|} - \inf_{\substack{\varphi \in \ell^2(\mathbb{C}) \\ \|\varphi\| = 1}} \langle \varphi | h^\varphi \varphi \rangle \leqslant 0$$

If $\left\|\gamma_d^{(1)}(0) - p_\varphi(0)\right\|_{S^1} = \mathcal{O}\left(\frac{1}{d}\right)$, then $\forall t \in \mathbb{R}_+$, $\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{S^1} \lesssim e^{Cte^{Ct}\sqrt{\ln(d)} - \ln(d)} \underset{d \to \infty}{\longrightarrow} 0$

Convergence of the order parameter: Use $a \leqslant \mathcal{N} + 1$ and insert a $\mathcal{N}$-cut-off:

$$\begin{aligned}
\left|\mathrm{Tr}\left(\gamma_d^{(1)} a\right) - \mathrm{Tr}\,(p_\varphi a)\right| &\leqslant \left\|\left(\gamma_d^{(1)} - p_\varphi\right) a\right\|_{S^1} = \left\|\left(\gamma_d^{(1)} - p_\varphi\right) a\,(\mathcal{N}+1)^{-1}(\mathcal{N}+1)\right\|_{S^1} \\
&\leqslant \left\|\left(\gamma_d^{(1)} - p_\varphi\right) \underbrace{a\,(\mathcal{N}+1)^{-1}(\mathcal{N}+1)\,\mathbb{1}_{\mathcal{N}<M}}_{\leqslant M}\right\|_{S^1} + \left\|\left(\gamma_d^{(1)} - p_\varphi\right) \underbrace{a\,(\mathcal{N}+1)^{-1}}_{\leqslant 1}(\mathcal{N}+1)\,\mathbb{1}_{\mathcal{N}\geqslant M}\right\|_{S^1} \\
&\leqslant M\left\|\gamma_d^{(1)} - p_\varphi\right\|_{S^1} + \underbrace{\mathrm{Tr}\left(\gamma_d^{(1)}(\mathcal{N}+1)\mathbb{1}_{\mathcal{N}\geqslant M}\right) + \mathrm{Tr}\left(p_\varphi(\mathcal{N}+1)\mathbb{1}_{\mathcal{N}\geqslant M}\right)}_{\to 0 \text{ when } M \to \infty \text{ since the particle numbers are conserved}}
\end{aligned}$$

Any choice of $M \gg 1$ such that $M\left\|\gamma_d^{(1)} - p_\varphi\right\|_{S^1} \ll 1$ as $d \to \infty$ is sufficient to prove that $\left\|\left(\gamma_d^{(1)} - p_\varphi\right) a\right\|_{S^1} \underset{d \to \infty}{\longrightarrow} 0$

References

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